THE SOLUTION OF THE EQUATION OF MAGNETO-HYDRODYNAMICS DESCRIBING THE ONE-DIMENSIONAL AXISYMMETRICAL MOTION OF A GAS IN A GRAVITATIONAL FIELD

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We consider the one-dimensional axisymmetrical unsteady motion of an electrically conducting perfect gas, subject to the force of Newtonian gravitation. We will assume the conductivity of the gas to be infinite, the magnetic field perpendicular to the trajectory of the gas molecules, and the viscosity and thermal conductivity negligible.

The magnetic force lines may be: (1) straight and parallel to the axis of symmetry; (2) concentric circles with centers on the axis of symmetry; (3) helical lines.

Let H_z be the component of the intensity vector along the axis of symmetry, and H_ϕ the transverse component of the same vector. We introduce the notation $h_z = H_z^2\, 1/8\pi$, $h_\phi = H_\phi^2\, 1/8\pi$. Then in case (1) above we will have $h_z \neq 0$, $h_\phi = 0$; in case (2), $h_z = 0$, $h_\phi \neq 0$; and in case (3), $h_z \neq 0$ and $h_\phi \neq 0$.

The equations of magneto-hydrodynamics for these cases may be written in the form

$$\rho \frac{dv}{dt} = -\frac{\partial}{\partial r}(p + h_\varphi + h_z) - \frac{2}{r}(h_\varphi + Gm\rho), \qquad \frac{d}{dt}\frac{h_\varphi}{r^2\rho^2} = 0, \qquad \frac{d}{dt}\frac{h_z}{\rho^2} = 0,$$
$$\frac{d\rho}{dt} = -\rho\left(\frac{\partial v}{\partial r} + \frac{v}{r}\right), \qquad \frac{dm}{dr} = 2\pi\rho r, \qquad \frac{d}{dt}\frac{p}{\rho^\gamma} = 0 \qquad (1)$$

Here r is the Euler coordinate of the particle, t the time, v the velocity, ρ the density, p the pressure, m the mass, G the gravitational constant, and $\gamma > 1$ the Poisson coefficient. We shall assume that the velocity v of the gas particle is proportional to its distance r from the axis of symmetry. That is,

$$v = \frac{r}{\mu(t)} \mu'(t) \tag{2}$$

where $\mu(t)$ is an arbitrary function of time, and $\mu'(t)$ is its derivative.

The linear dependence (2) of the velocity on the radius has been used previously in the work of Sedov (1, 2, 3), Lidov [4], Kulikovskii [5], Iavorskaia [6], and Korobeinikov [7].

It can be established by direct substitution that system (1) has the following particular solution:

$$\begin{gathered} v = \frac{r}{\mu(t)} \mu'(t), \qquad \rho = \frac{P'(\xi)}{r\mu}, \qquad p = [b_1 P(\xi) + b_2] \mu^{-2\gamma} \\ h_\varphi = \frac{1}{r^2} \left\{ b_4 + b_3 \left[\frac{r^2}{\mu^2} P(\xi) - 2P_1(\xi) \right] - 2\pi G P^2(\xi) \right\} \\ h_z = [b_5 P(\xi) + b_6] \mu^{-1}, \qquad m = 2\pi P(\xi) \end{gathered} \tag{3}$$

where $\mu(t)$ satisfies the equation

$$t = \pm \int \frac{d\mu}{\sqrt{f(\mu)}}, \qquad f(\mu) = \frac{b_1}{\gamma - 1} \mu^{2(1-\gamma)} - 2b_3 \ln \mu + b_5 \mu^{-2} + b_7 \tag{4}$$

Here b_1 and b_7 are arbitrary constants and $P(\xi)$ an arbitrary function such that its derivative $P'(\xi) > 0$. The function $P_1(\xi)$ is related to $P(\xi)$ by the relation

$$P_1'(\xi) = \xi P(\xi) \qquad \left(\xi = \frac{r}{\mu}\right)$$

ξ is the Lagrange coordinate. In the particular case where $h_\phi = 0$, the function $P(\xi)$ has the form $P(\xi) = a_1 \xi^2$, where a_1 is a constant. This case, and also the case where $h_z = 0$, and $P(\xi) = a_2 \xi^2 + a_3$, where a_2 and a_3 are arbitrary constants, were derived and investigated by Iavorskaia [6].

A solution was obtained for case $G = 0$ by Kulikovskii, who investigated it in detail, for case $h_z = 0$. We proceed to a consideration of the behaviour of function $f(\mu)$. Its form depends on the magnitudes of the constants b_1, b_3, b_5 and b_7; and it distinguishes different types of motion of the gas.

Case $b_3 > 0$ gives the motion described in reference [6]. Also, if $b_3 = 0$, function $f(\mu)$ has the same form as in reference [5]. Consequently, in that case, the same motion of the gas is possible as was considered in reference [5]. For $b_3 < 0$, $b_5 \neq 0$, the following cases are possible:

$$
\text{I}\begin{cases} b_1 \geqslant 0, & b_5 > 0 \\ b_1 > 0, & b_5 < 0, \quad \gamma > 2 \\ b_1 < 0, & b_5 > 0, \quad \gamma < 2 \\ b_1 + b_5 > 0, & \gamma = 2 \end{cases} \qquad \text{II}\begin{cases} b_1 \leqslant 0, & b_5 < 0, \quad b_7 > 0 \\ b_1 > 0, & b_5 < 0, \quad \gamma < 2 \\ b_1 < 0, & b_5 > 0, \quad \gamma > 2 \\ b_1 + b_5 \leqslant 0, & b_7 > 0, \quad \gamma = 2 \end{cases}
$$

In case I the type of motion depends on the number of roots of function $f(\mu)$ and their distribution.

(1) If for function $f(\mu)$ there is one double root $\mu_0 = 1$, the gas will exist in the state of unstable equilibrium.

(2) If the double root $\mu_0 \neq 1$, a limit motion of the gas can exist, that is, expansion (or compression) of the gas at infinite time up to some finite volume of radius $r = \xi\mu_0$. For this, $v \to 0$ as $t \to \infty$.

If the limit motion does not exist, there will also be the following possibilities

(2a) For $\mu_0 > 1$ and $v_0 < 0$ complete compression of the gas takes place to the axis of symmetry, in finite time.

(2b) For $\mu_0 > 1$ and $v_0 > 0$, the expanding gas reaches the limiting distance $r = \xi\mu_0$ from the axis of symmetry; and thereafter, under the action of the force, a reverse motion starts; that is, a complete compression of the gas takes place to the axis of symmetry, at infinite time. If in this case we assume that at the moment of constriction of the gas the velocity changes discontinuously, we have an example of pulsing periodic motion of the gas.

(2c) For $\mu_0 < 1$ and $v_0 > 0$ a complete scattering of the gas would take place in infinite time, with the velocity $v \to \infty$ as $t \to \infty$.

(2d) For $\mu_0 < 1$ and $v_0 < 0$ the gas is initially compressed to some volume with radius $r = \xi\mu_0$, and thereafter is completely dispersed.

(3) Function $f(\mu)$ does not have roots. Then for $v_0 < 0$ the motion is analogous to case (2a), and for $v_0 > 0$, to case (2c).

(4) Function $f(\mu)$ has two roots μ_1 and μ_2. If $\mu_1 < \mu_2 < 1$, there will be complete dispersal as in cases (2c) and (2d). If $1 < \mu_1 < \mu_2$, there will be complete condensation, as in cases (2a) and (2c).

In case (2) we have $f'(\mu) > 0$. Complete dispersal of the gas occurs as in cases (2c) and (2d).

For $b_3 = 0$ and $b_5 = 0$, motion of the types (2a), (2b), (2c), (2d) and (3) are possible. In the cases when the gas is dispersed, the velocity $v \to \xi\sqrt{b_7}$ as $t \to \infty$. Detailed investigation was carried out in the book by Sedov [1].

If $b_3 = 0$, $b_5 \neq 0$, we have the following cases:

$$\text{III} \begin{cases} b_1>0, & b_5<0, & b_7>0, & \gamma>2 \\ b_1<0, & b_5>0, & b_7>0, & \gamma<2 \end{cases} \qquad \text{IV} \begin{cases} b_1 \geqslant 0, & b_5>0, & b_7 \geqslant 0 \\ b_1+b_5>0, & b_7 \geqslant 0, & \gamma=2 \end{cases}$$

$$\text{V} \begin{cases} b_1>0, & b_5<0, & b_7 \geqslant 0, & \gamma<2 \\ b_1<0, & b_5>0, & b_7 \geqslant 0, & \gamma>2 \\ b_1 \leqslant 0, & b_5<0, & b_7>0 \\ b_1+b_5<0, & b_7>0, & \gamma=2 \end{cases} \qquad \text{VI} \begin{cases} b_1>0, & b_5<0, & b_7 \leqslant 0, & \gamma>2 \\ b_1<0, & b_5>0, & b_7 \leqslant 0, & \gamma<2 \\ b_1 \geqslant 0, & b_5>0, & b_7<0 \\ b_1+b_5>0, & b_7<0, & \gamma=2 \end{cases}$$

$$\text{VII} \begin{cases} b_1>0, & b_5<0, & b_7<0, & \gamma<2 \\ b_1<0, & b_5>0, & b_7<0, & \gamma>2 \end{cases}$$

In case 3 the same motion is possible as in case (1), except that for complete dispersal, $v \to \xi\sqrt{b_7}$. In case (4) the motion exists just as in case (3), except that $v \to \xi\sqrt{b_7}$ as $t \to \infty$.

In case (5) we have complete dispersal of the gas (see (2c), (2d)) but $v \to \xi\sqrt{b_7}$, as $t \to \infty$.

In case (6) complete condensation of the gas takes place in a finite period of time, as in cases (2a) and (2c).

In case (7) a periodic oscillation of the gas takes place with a period depending on the quantities b_1, b_3, b_5 and b_7. In special cases, where $f(\mu)$ has double roots, we have stable equilibrium of the gas.

From the research presented here, it is evident that in the presence of gravitational force and the magnetic field with spiral lines fo force, complete dispersal of the gas is shown to be impossible, if $b_3 > 0$. Also if $b_3 \leqslant 0$, dispersal is shown to be in some cases possible and in some cases not possible. In the particular case when the magnetic field direction is along the axis of symmetry, dispersal of the gas is not possible [6], because $b_3 > 0$ for $h_\phi = 0$. In the other particular case when the magnetic force lines are essentially closed concentric circles ($h_z = 0$), for $b_3 \leqslant 0$ dispersal is possible, and for $b_3 > 0$ it is not.

We remark in conclusion that in the case where $\gamma = 2$, equation (1) permits of an exact particular solution, containing two arbitrary functions:

$$v = \frac{r}{\mu(t)}\mu'(t), \qquad \rho = \frac{P'(\xi)}{r\mu}, \qquad \rho = \Pi(\xi)\mu^{-4},$$

$$h_z = [c_1 P(\xi) - \Pi(\xi) + c_2]\mu^{-4}$$

$$h_\varphi = \frac{1}{r^2}\{c_3[\xi^2 P(\xi) - 2P_1(\xi)] - 2\pi G P^2(\xi) + c_4\}$$

$$m = 2\pi P(\xi), \qquad P_1'(\xi) = \xi P(\xi)$$

$$f(\mu) = \left(\frac{d\mu}{dt}\right)^2 = c_1\mu^{-2} - 2c_3 \ln \mu + c_5$$

where c_1, c_2, ..., c_5 are arbitrary constants, $P(\xi)$ and $\Pi(\xi)$ are arbitrary functions. A solution with two arbitrary functions can also be obtained for $\gamma = 1$.

Note added in proof. After this paper had been submitted for publication, the author discovered that a solution analogous to (3) had also been published in a recent paper by McVittie [8]. Analysis of the possible motions of the gas is missing in reference [8].

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